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Analytic capacities of finite sequences in the unit disc in Besov spaces

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Abstract

This study drives new estimate on analytic capacities of finite sequences in the unit disc in Besov spaces with zero smoothness, for a range of parameters, are optimal. The work is motivated both from the perspective of complex analysis by the description of sets of zero/uniqueness, and from the one matrix analysis / operator theory by estimates on norms of inverses.

Keywords: Besov space of analytic functions; Blaschke products; Analytic capacities; Functional calculus

1. Introduction

Let $\mathbb{D} = \{(\omega - \epsilon) \in \mathbb{D} : |\omega - \epsilon| < 1\}$ be the open unit disk, let $\mathbb{T} = \{(\omega - \epsilon) \in \mathbb{D} : |\omega - \epsilon| = 1\}$ be its boundary and $\mathbb{D}_* = \mathbb{D}\backslash\{0\}$. We denote by $\mathcal{H}ol(\mathbb{D})$ the space of analytic functions on $\mathbb{D}$, equipped with the topology of local uniform convergence. Let X be a Banach space that is continuously contained in $\mathcal{H}ol(\mathbb{D})$ and that contains the polynomials. Given a finite sequence $\mu = \ltimes_1, \cdots\cdots, \ltimes_2 \in \mathbb{D}_*^N$, Nikolski [10] defined the X −zero capacity of μ as

$$\mathrm{cap}_X(\mu) = \inf\{|f^2|_X : f^2(0) = 1\,, f^2\,|\mu = 0\},$$

where $f^2\,|\mu = 0$ means that $f^2(\ltimes_i) = 0$ for all $i = 1, \cdots\cdots, N$ taking into account possible multiplicities. Namely, if $\mu = (\ltimes_1, \cdots\cdots, \ltimes_1, \ltimes_2, \cdots\cdots, \ltimes_2, \cdots\cdots, \ltimes_s, \cdots\cdots, \ltimes_s) \in \mathbb{D}^N$, where each $\ltimes_i$ is repeated according to its multiplicity $x_i \geq 1$, then $f^2\,|\mu = 0$ means that

$$f^2(\ltimes_i) = \acute{f}^2(\ltimes_i) = f^{"2}(\ltimes_i) = \cdots\cdots = f^{2^{(x_i-1)}}(\ltimes_i) = 0, \qquad i = 1, \cdots\cdots, s$$

Anton Baranov, Michael Hartz, Hgiz Kayumov, Rachid Zarouf [23]. The researcher intends to make few specific changes.

1.1. Motivation from complex analysis: sets of zeros/uniqueness

Form the point of view of complex analysis, the $X-$zero capacities are closely related to the problem of characterizing uniqueness sets for the function space X; here μ is said to be a uniqueness sets for X if $f^2 \in X$, $f^2|\mu = 0 \Longrightarrow f^2 = 0$. Following [10], assume that the function space X satisfies the following Fatou property: if $f_n^2 \in X$, $\sup_\epsilon \|f_\epsilon^2\|_X < \infty$ and $\lim_{\epsilon\to\infty} f_\epsilon^2(\omega - \epsilon) = f^2(\omega - \epsilon)$ for $(\omega - \epsilon) \in \mathbb{D}$, then $f^2 \in X$. Then it is not hard to see that an infinite sequence $\mu = (\ltimes_i)_{i\geq 1} \in \mathbb{D}_*^\infty$ is a uniqueness for X if and only if

$$\sup_N\{\mathrm{cap}_X(\mu_N)\} = \infty, \qquad ..... \ (1)$$

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Where $\mu_N = (\ltimes_i)_{i=1}^N$ is the truncation of μ of order N. For example, let X be the algebra H^∞ of bounded holomorphic functions in $\mathbb{D}$ endowed with the norm $\|f^2\|_{H^\infty} = \sup_{\zeta\in\mathbb{D}} |f^2(\zeta)|$. It is known [10, Theorem 3.12] that given $\mu_N = (\ltimes_i)_{i\geq 1} \in \mathbb{D}_*^\infty$,

$$\text{cap}_{H^\infty}(\mu_N) = \frac{1}{\prod_{i=1}^N |\ltimes_i|}. \quad \text{..........(2)}$$

$$A = A_{\mu N} = \prod_{i=1}^N \frac{(\omega-\epsilon)-\ltimes_i}{1-\bar{\ltimes}_i},$$

The finite Blaschke product associated with μ_N, observe that the right-hand side in (2) is achieved by the test function $f^2 = A/A(0)$, which is admissible for the conditions in the infimum defining the capacity of μ_N. Thus, an application of the above criterion (1) leads to the well-known Blaschke condition: an infinite sequence $\mu_N = (\ltimes_i)_{i\geq 1} \in \mathbb{D}_*^\infty$, is a uniqueness sequence for H^∞ if and only if

$$\sum_{i\geq 1} (1 - |\ltimes_i| = \infty,$$

1.2. Motivation in operator theory/matrix analysis

Let T be an invertible operator acting on a Banach space or and $N \times N$ invertible matrix with complex entries acting on $\mathbb{C}^N$ equipped with some norm. The researcher seeks upper bounds on the norm of the inverse T^{-1}. Assume that the minimal polynomial of T is given by

$$m(\omega-\epsilon) = m_T(z) = \prod_{i=1}^N ((\omega-\epsilon) - \ltimes_i),$$

Where $\mu_N = (\ltimes_i)_{i\geq 1} \in \mathbb{D}_*^\infty$ and the researcher assumed for simplicity that $\deg m_T = N$. Following [10], assume that our Banach space $X \subset \mathcal{H}ol(\mathbb{D})$ is an in fact an algebra, and write $B = X$. Assume further that

(1) T admits a C −functional calculus on B, i.e. there exists a bounded homomorphism $f^2 \longmapsto f^2(T)$ extending the polynomial functional calculus and a constant $C > 0$ such that

$$\|f^2(T)\| \leq C\|f^2\|_B, \qquad f^2 \in B;$$

(2) the shift operator $S : f^2 \mapsto (\omega-\epsilon)f^2$, the backward shift operator $S^* : f^2 \mapsto \frac{f^2 - f^2(0)}{\omega-\epsilon}$ and the generalized backward shift operators $f^2 \mapsto \frac{f^2 - f^2(\ltimes)}{(\omega-\epsilon)-\ltimes}$ are bounded on B for all $\ltimes \in \mathbb{D}$.

There assumptions are mild and satisfied by all the algebras B considered below. Noticing that the analytic polynomial $1 + \epsilon = \frac{m(0)-m}{(\omega-\epsilon)m(0)}$ interpolates the function $\frac{1}{\omega-\epsilon}$ on μ the researcher observes that

$$T^{-1} = (1+\epsilon)(T) = (1+\epsilon) + mh)(T),$$

For any $h^2 \in B$. Applying assumption (1) to the above operator the researcher obtains

$$\|T^{-1}\| \leq C\|(1+\epsilon) + mh^2\|_B$$

and taking the infimum over all $h^2 \in B$ and using our assumption on B, the researcher gets

$$\|T^{-1}\| \leq C\ inf\left\{\|g^2\|_A : g^2|\mu = (1+\epsilon)|\mu = \frac{1}{\omega-\epsilon}|\mu\right\}. \quad \text{........ (3)}$$

Now, if $f^2 \in B$ satisfies $f^2(0) = 1$ and $f^2|\mu = 0$, then $f^2 := S^*(1-f^2) = S^*(f^2)$ is admissible for the last infimum, and so

$$\|T^{-1}\| \leq C\|S^*\|_{B\to B}\, \text{cap}_B(\mu). \quad \text{......... (4)}$$

In particular (3) and (4) are applied (among other situations) in [10] to the cases of:

- Hilbert space contractions, B the disc algebra and $C = 1$;
- Banach space contractions, B the Wiener algebra of absolutely convergent Taylor/Fourier series,

$$B = W = \left\{ f^2 = \sum_{k\geq 0} \widehat{f^2}(k) z^k \in \mathcal{H}ol(\mathbb{D}) : \|f^2\|_W = \sum_{k\geq 0} \left|\widehat{f^2}(k)\right| \leq \infty \right\},$$

And once again $C = 1$;

- Tadmor- Ritt type matrices or power- bounded matrices on Hilbert spaces and B the Besov algebra

$$B = \mathcal{A}^0_{\infty,1} = \left\{ f^2 \in \mathcal{H}ol(\mathbb{D}) : \|f^2\|_{\mathcal{A}^0_{\infty,1}} = |f^2(0)| + \int_0^1 \left\| f'^2_{(1+\epsilon)} \right\|_{L^\infty(\mathbb{T})} d(1+\epsilon) < \infty \right\},$$

where

$$f^2_{(1+\epsilon)}(\zeta) = f^2(1+\epsilon)\zeta), \zeta \in \mathbb{T}.$$

Upper estimate on $cap_X(\mu)$ where X is a general Besov space $\mathcal{A}^{1+\epsilon}_{(1+\epsilon),(\frac{1}{2}+3\epsilon)}$ $s \geq 0, (1+\epsilon), (\frac{1}{2}+3\epsilon) \in [1,\infty]^2$, see below for their definition. We also relate the special case $(1+\epsilon), (\frac{1}{2}+3\epsilon) \in (1,\infty)$ to applications in operator theory/ matrix analysis and especially to Schäffer's question on norms of inverses. The researcher formulate the main results see[23].Theorem 2, which corresponds to the special case $(1+\epsilon), \left(\frac{1}{2}+3\epsilon\right) \in (1,\infty)$, exhibits an explicit sequence $\mu^\star$ the researcher derives a quantitative lower bound on $\mathrm{cap}_{\mathcal{A}^0_{\infty,1}}(\mu^\star)$ and thereby almost prove the sharpness of Nikolski's upper bound in the case. Theorem 3 improves Nikolski's upper bounds on $cap_{\mathcal{A}^0_{(1+\epsilon),(\frac{1}{2}+3\epsilon)}}(\mu)$ for a range of parameters, while in Theorem 4 the sharpness of these new bounds is discussed.

2. Known results and open questions

2.1. Capacities in Besov spaces

The case where X is an analytic Besov spaces $X = \mathcal{A}^{1+\epsilon}_{(1+\epsilon),(\frac{1}{2}+3\epsilon)}$ is considered in [10]. Let $\epsilon \geq 0, 0 \leq \epsilon$, $\epsilon \leq \infty$, and let

$$\mathcal{A}^{1+\epsilon}_{(1+\epsilon),(\frac{1}{2}+3\epsilon)} = \left\{ f^2 \in \mathcal{H}ol(\mathbb{D}) : \|f^2\|_{\mathcal{A}^{1+\epsilon}_{(1+\epsilon),(\frac{1}{2}+3\epsilon)}} = \left(\int_0^1 ((\epsilon)^{1+\frac{\epsilon}{\frac{1}{2}}+3\epsilon} \|f^{(2+4\epsilon)}_{1-\epsilon}\|_{L^{(1+\epsilon)}(\mathbb{T})})^{\frac{1}{2}+3\epsilon} d(1-\epsilon) \right)^{1/\frac{1}{2}+3\epsilon} < \infty \right\},$$

where $f^{2+4\epsilon}_{1-\epsilon}(\zeta) = f^{2+4\epsilon}_{1-\epsilon}\big((1-\epsilon)\zeta\big)$, $(1+2\epsilon)$ being a nonnegative integer such that $\epsilon > 0$ (the choice of $(1+2\epsilon)$ is not essential and the norms for different ϵ are equivalent). The researcher obvious modification for $\epsilon = \infty$. The space $\mathcal{A}^{1+\epsilon}_{(1+\epsilon),(\frac{1}{2}+3\epsilon)}$ equipped with the norm

$$\|f^2\|_{\mathcal{A}^{1+\epsilon}_{(1+\epsilon),(\frac{1}{2}+3\epsilon)}} = \sum_{k=0}^{2\epsilon} \left| f^{(2k)}(0) \right| + \|f^2\|^*_{\mathcal{A}^{1+\epsilon}_{(1+\epsilon),(\frac{1}{2}+3\epsilon)}},$$

Is a Banach space. The researcher refers to [5,13,20] for general properties of Besvo spaces. Note that for $0 \leq \epsilon < \infty$ the researcher has $f^2_{1-\epsilon} \to f^2$ in the norm of $\mathcal{A}^{1+\epsilon}_{(1+\epsilon),(\frac{1}{2}+3\epsilon)}$ as $\epsilon \to 0$. The researcher deals with Besov spaces with zero smoothness $\epsilon = -1$. The researcher takes $\epsilon = 0$ and

$$\|f^2\|^*_{\mathcal{A}^{1+\epsilon}_{(1+\epsilon),(\frac{1}{2}+3\epsilon)}} = \left(\int_0^1 ((\epsilon)^{-\frac{1}{2}+3\epsilon}\ \|f'_{1-\epsilon}\|^{\frac{1}{2}+3\epsilon}_{L^{1+\epsilon}(\mathbb{T})} d(1-\epsilon)\right)^{1/\frac{1}{2}+3\epsilon}, \qquad 0 < \epsilon < \infty,$$

$$\|f^2\|^*_{\mathcal{A}^0_{(1+\epsilon),\infty}} = \sup_{-1<\epsilon<0} (\epsilon)\ \|f'_{1-\epsilon}\|^{\frac{1}{2}+3\epsilon}_{L^{1+\epsilon}(\mathbb{T})}.$$

Note that $\mathcal{A}^0_{\infty,\infty}$ coincides with the classical Bloch space.

It is shown [10, Theorem 3.26] that given $0 \le \epsilon$, $\epsilon \le \infty$, $\epsilon > -1$ and $\mu \in \mathbb{D}^N_*$ the following upper estimate holds

$$cap_{\mathcal{A}^{1+\epsilon}_{(1+\epsilon),(\frac{1}{2}+3\epsilon)}}(\mu) \le c \frac{N^{1+\epsilon}}{\prod_{i=1}^N |\ltimes_i|},$$

where $c = c(1+\epsilon, \frac{1}{2}+3\epsilon)$, and that if $\epsilon = -1$ then

$$cap_{\mathcal{A}^{1+\epsilon}_{(1+\epsilon),(\frac{1}{2}+3\epsilon)}}(\mu) \le c \frac{(\log N)^{\frac{1}{\frac{1}{2}}+3\epsilon}}{\prod_{i=1}^N |\ltimes_i|}, \qquad \dots\dots (5)$$

where $c > 0$ is numerical constant. It is also shown that for $\epsilon > -1$ these estimates are asymptotically sharp the researcher gives and show main results (see [23]) and see [10, Theorem 3.31]: there exist constants $c = c(1+\epsilon, 1+\epsilon, \frac{1}{2}+3\epsilon) > 0$ and $k = k(1+\epsilon, 1+\epsilon, \frac{1}{2}+3\epsilon) > 0$ such that for any $\mu = (\ltimes_1, \cdots\cdots, \ltimes_N) \in \mathbb{D}^N_*, \epsilon > -1$, $0 \le \epsilon$, $\epsilon \le \infty$,

$$cap_{\mathcal{A}^{1+\epsilon}_{(1+\epsilon),(\frac{1}{2}+3\epsilon)}}(\mu) \le c \frac{N^{1+\epsilon}}{\prod_{i=1}^N |\ltimes_i|} \ (1 + K - \prod_{i=1}^N (1 + |\ltimes_i|).$$

The sharpness of the upper bound in (5) is left as an open question in [10].

2.2. Norms of inverses and Schäffer′s question

Let $\|\cdot\|$ denote the operator norm induced on $\mathcal{M}_{N,}$ the space of complex $N \times N$ matrices , by a Banach space norm on $\mathbb{C}^N$. What is the smallest constant $\mathcal{S}_N$ so that

$$|\det T| \cdot \|T^{-1}\| \le \mathcal{S}_N \|T\|^{N-1}$$

holds for any invertible matrix $T \in \mathcal{M}_N$ and any operator norm $\|\cdot\|$ Schäffer′s [17, Theorem 3.8] prove that

$S_N = \sqrt{eN}$,

but he conjectured that S_N should in fact be bounded, as it is the case for Hilbert space. This conjecture was disproved by E.Gluskin, M.Meyer, and A. Pajor [7]. Later, Queffe′lec [15] showed that the $\sqrt{N}$ bound is essentially optmail for arbitrary Banach spaces, but both arguments are non-constructive. An explicit construction giving a $\sqrt{N}$ lower bound was recently given in [19]. For a detailed account on the history of Schäffer′s question, the reader is referred to [19]. A key tool in the works cited above is the equality.

$$S_N = \sup_{(\ltimes_1,\cdots,\ltimes_N) \in \mathbb{D}^N} \prod_{i=1}^N |\ltimes_i| (\mathrm{cap}_W(\ltimes_1, \cdots, \ltimes_2) - 1), \quad \dots\dots\dots\dots(6)$$

Due to Gluskin, Meyer and pajor. It connects Schäffer′s question to capacity in the Wiener algebra and shows that (4) is essentially sharp.

It is natural to consider Schäffer′s question for operator classes different from Hilbert or Banach space contractions see [10], the researcher considers the following classes, which admit a Besov $\mathcal{A}^0_{\infty,1-}$ functional calculus.

(1) power bounded operators on Hilbert space, i.e. operators T on Hilbert space satisfying

$$\sup_{k \geq 0} \|T^K\| = C_{\frac{1}{2}+\frac{5}{2}\epsilon+\epsilon^2} < \infty.$$

Peller [14] proved that $\|f^2 T\| \leq k_G C^2_{\frac{1}{2}+\frac{5}{2}\epsilon+\epsilon^2} \|f^2\|_{\mathcal{A}^0_{\infty,1}}$ for every analytic polynomial f^2, where k_G is the Groythendieck constant. Combining (4) with Nikolski's upper estimate (5) for $\epsilon = 0$, the researcher obtains the upper bounds

$$\|T^{-1}\| \leq c_1 . \operatorname{cap}_{\mathcal{A}^0_{\frac{1}{2}+\frac{5}{2}\epsilon+\epsilon^2}}(\ltimes_1, \cdots, \ltimes_N) < c_3 \frac{k_G C^2_{\frac{1}{2}+\frac{5}{2}\epsilon+\epsilon^2} \log N}{\prod_{i=1}^N |\ltimes_i|}, \quad \ldots\ldots\ldots \quad (7)$$

where $c_1 > 0$ is an absolute constant and $(\ltimes_i)_{I=1}^N$ is the sequence of eigenvalues of T.

(2) Tadmor-Ritt operators on Banach space, i.e. operators T acting on a Banach space and satisfying the resolvent estimate

$$\sup_{|\zeta| > 1} |\zeta - 1| \|(\zeta - T)^{-1}\| = C_{TR} < \infty.$$

According to P. Vitse's functional calculus [22, Theorem 2.5] the researcher has $\|f^2 T\| \leq 300 C^5_{TR} \|f^2\|_{\mathcal{A}^0_{\infty,1}}$ for every analytic polynomial f^2, and the following the same reasoning as above this yields

$$\|T^{-1}\| \leq c_2 . \operatorname{cap}_{\mathcal{A}^0_{\infty,1}}(\ltimes_1, \cdots, \ltimes_2) < c_2 \frac{300 C^5_{TR} \log N}{\prod_{i=1}^N |\ltimes_i|}, \quad \ldots\ldots \quad (8)$$

where $c_2 > 0$ is an absolute constant. The researcher thanks to work of Schwenninger [18], the dependence on C_{TR} can be improved from C^5_{TR} to $C_{TR}(\log C_{TR} + 1)$.

The sharpness of the right-hand side in (7) and (5) is an open question both from the point of view of operators / matrices and from the one of capacities. Note that the researcher has strict inclusions:

$$W \subset \mathcal{A}^0_{\infty,1} \subset H^\infty \quad \ldots\ldots (9)$$

see [5,13,11]. Observe that $\mathcal{A}^0_{\infty,1}$ is actually contained in the disc algebra. From the perspective of capacities (9) implies that for any sequence $\mu = (\ltimes_1, \cdots, \ltimes_N) \in \mathbb{D}^N_*$ the researcher has

$$\operatorname{cap}_{H^\infty}(\mu) \leq c_3 \operatorname{cap}_{\mathcal{A}^0_{\infty,1}}(\mu) \leq c_4 \operatorname{cap}_W(\mu) \quad \ldots\ldots (10)$$

where $c_3 . c_4 > 0$ are absolute constants. Observe that in view of (10) and (6) any sequence $\mu \in \mathbb{D}^N_*$ such that $\prod_{i=1}^N |\ltimes_i| . \operatorname{cap}_{\mathcal{A}^0_{\infty,1}}(\mu)$ grows unboundedly in N will automatically give a counterexample to Sch äffer's original question.

3. Main results

The researcher uses the following standard notation. For two positive functions f^2, g^2 the researcher says that f^2 is dominated by g^2, denoted by $f^2 \lesssim g^2$, if there is a constant $c > 0$ such that $f^2 \leq c g^2$ for all admissible variables . The researcher says that f^2 and g^2 are comparable, denoted by $f^2 \asymp g^2$, if both $f^2 \lesssim g^2$ and $g^2 \lesssim f^2$.

The main goals of this paper are to

- Prove an example of a sequence $\mu^* = (\ltimes_1, \cdots, \ltimes_N) \in \mathbb{D}^N_*$ such that $\prod_{i=1}^N |\ltimes_1| \cdot cap_{\mathcal{A}^0_{\infty,1}}(\mu^*)$ almost approaches Nikolski's upper bound $\log N$.
- Improve Nikolski's upper bound (5) on $\prod_{i=1}^N |\ltimes_1| \cdot cap_{\mathcal{A}^0_{\infty,1}}(\mu)$ identifying three regions of $(1+\epsilon, \frac{1}{2} + 3\epsilon) \in [1, \infty]^2$ with a different behavior of this quantity (see theorem 3 below). For all $(1+\epsilon, \frac{1}{2} + 3\epsilon)$ with $\epsilon \neq \infty$ our

estimates give a smaller growth than the estimate in [10], and for a range of parameters, namely for $1 < \frac{1}{2} + 3\epsilon < 1 + \epsilon < \infty$ and $\epsilon \geq 1$, they are best possible.

3.1. A lower estimate on $\mathrm{cap}_{\mathcal{A}^0_{\infty,1}}(\mu)$

Our approach to bounding $\mathrm{cap}_{\mathcal{A}^0_{\infty,1}}(\mu)$ from below uses duality. To estimate $\mathrm{cap}_{\mathcal{A}^0_{\infty,1}}(\mu)$ from below, the researcher estimates the Besov seminorm in A_(1,∞)^0 of finite Blaschke products from above. The key inequality, which will be proved in Lemma 1 below, is

$$\mathrm{cap}_{\mathcal{A}^0_{\infty,1}}(\mu) \gtrsim \frac{1}{\prod_{i=1}^{N}|\ltimes_i|}\frac{1-\prod_{i=1}^{N}|\ltimes_i|^2}{\|\mathcal{A}\|^*_{\mathcal{A}^0_{1,\infty}}}, \qquad \ldots\ldots. (11)$$

where $\mu = (\ltimes_1, \cdots, \ltimes_N)$ is an arbitrary sequence in $\mathbb{D}^N_*$, and $\mathcal{A} = \mathcal{A}_\mu$ is the finite Blaschke product associated to μ. To conclude the researcher considers $\epsilon > 1$ and for $k = 1, \cdots\cdots, 1+\epsilon$ the researcher puts

$$\mu_k = (1-\epsilon_k^{(1+\epsilon)} e^{2i\pi j/2^k})_{j=1}^{2^k} \in \mathbb{D}^{2^k}_*, \qquad 1-\epsilon_k^{(1+\epsilon)} = \left(1-\frac{1}{1+\epsilon}\right)^{2^{-k}}.$$

The researcher puts $N = \sum_{k=1}^{1+\epsilon} 2^k \asymp 2^{1+\epsilon}$ and define the sequence $\mu^* = (\ltimes_1, \cdots\cdots, \ltimes_N) \in \mathbb{D}^N_*$ by

$$\mu^* = \left(\mu_1, \mu_2, \cdots, \mu_{(1+\epsilon)}\right). \qquad \ldots\ldots\ldots (12)$$

Denoting by $\mathcal{A}^*$ the Blaschke product associated with μ^* the researcher has

$$\mathcal{A}^*(\omega-\epsilon) = \prod_{k=1}^{1+\epsilon} \frac{(\omega-\epsilon)^{2^k}-(1+\epsilon)}{1-(1+\epsilon)(\omega-\epsilon)^{2^k}}, \qquad \ldots\ldots\ldots. (13)$$

where $\epsilon = -\frac{1}{1+\epsilon}$. The researcher proves and the following results see[23].

Proposition 1. The Blaschke product $\mathcal{A}^*$ satisfies

$$\|\mathcal{A}^\star\|^*_{\mathcal{A}^0_{1,\infty}} \lesssim \frac{\log\log\mathrm{N}}{\log\mathrm{N}}. \qquad \ldots\ldots\ldots. (14)$$

Taking into account that $\prod_{j=1}^{N}|\ltimes_i| \leq e^{-1}$ and combining (11) with (14) the researcher obtains the foiiowing theorem.

Theorem 2. Let $\mu^\star = \mathbb{D}^N_*$ and $\mathcal{A}^\star$ be defined by (12) and (13). Then

$$\prod_{i=1}^{N}|\ltimes_i|\,.\,cap_{\mathcal{A}^0_{\infty,1}}(\mu^\star) \gtrsim \frac{\log\mathrm{N}}{\log\log\mathrm{N}}.$$

As a consequence regarding Schäffer's question, Theorem 2 implies (taking into account (10)) that

$$\prod_{i=1}^{N}|\ltimes_i|\,.\,cap_W(\mu^\star) \gtrsim \frac{\log\mathrm{N}}{\log\log\mathrm{N}}.$$

From this, following arguments in [19], one obtains another explicit counterexample to Schäffer's question,

3.1.1. Proof of Proposition 1

For simplicity the researcher write $\mathcal{A}$ instead of $\mathcal{A}^\star$ throughout the proof. Then $N = \deg\mathcal{A} \asymp 2^{1+\epsilon}$.

For the zeros $z_1, \cdots\cdots, z_2$ of $\mathcal{A}$ the researcher has

$$\prod_{j=1}^{N}|z_j| = (1+\epsilon)^{1+\epsilon} < e^{-1}.$$

For $z \in \mathbb{D}$. $|z| = 1-\epsilon$, the researcher has

$$|\mathcal{A}'(z)| \leq \sum_{k=1}^{1+\epsilon} 2^k(1-\epsilon)^{2^k-1} \ \frac{-2\epsilon-\epsilon^2}{|1-(1+\epsilon)z^{2k}|^2}. \quad \dots\dots (15)$$

Using that $\|(1-(1+2\epsilon)z^N\|_{H^2}^2 = \frac{1}{4}(-1-\epsilon^2)^{-1}$ for $\frac{-1}{2} \leq \epsilon < 0$, the researcher finds that $\epsilon \int_0^{2\pi}\left|\mathcal{A}'(1-\epsilon)e^{it})d(1+\epsilon)\right| \leq 2\pi \sum_{k=1}^{1+\epsilon} 2^k(1-\epsilon)^{2^{k-1}} \frac{-\epsilon^2-\epsilon^3}{1-(1+2\epsilon+\epsilon^2)(1-\epsilon)^{2^{k+1}}} \lesssim$

$$\tfrac{1}{1+\epsilon}\textstyle\sum_{k=1}^{1+\epsilon} 2^k(1-\epsilon)^{2^k-1} \frac{\epsilon}{1-(1+\epsilon)(1-\epsilon)^{2^k}}.$$

Let us first estimate this quantity for $1 \leq \epsilon \leq \frac{1}{2}$. In this case

$$\frac{1}{1+\epsilon}\sum_{k=1}^{1+\epsilon} 2^k(1-\epsilon)^{2^k-1}\frac{\epsilon}{1-(1+\epsilon)(1-\epsilon)^{2^k}} \lesssim \frac{1}{1+\epsilon}\sum_{k=1}^{1+\epsilon} 2^{k-2^k} \lesssim \frac{1}{1+\epsilon}.$$

Form now one the researcher assumes that $\epsilon = \frac{1}{2^{1+\epsilon}}$ where $\epsilon > 0$, and the researcher writes

$$\begin{aligned}&\frac{1}{1+\epsilon}\sum_{k=1}^{1+\epsilon} 2^k(1-\epsilon)^{2^k-1} \ \frac{\epsilon}{1-(1+\epsilon)(1-\epsilon)^{2^k}}\\ &= \frac{1}{1+\epsilon}\sum_{k=1}^{[1+\epsilon]} 2^k(1-\epsilon)^{2^k-1} \ \frac{\epsilon}{1-(1+\epsilon)(1-\epsilon)^{2^k}} + \frac{1}{1+\epsilon}\sum_{k=[1+\epsilon]+1}^{1+\epsilon} 2^k(1-\epsilon)^{2^k-1} \ \frac{\epsilon}{1-(1+\epsilon)(1-\epsilon)^{2^k}}\\ &= (1+\epsilon)_1 + (1+\epsilon)_2 .\end{aligned}$$

Since $(1-x)^{1+\epsilon} < e^{-(1+\epsilon)x}$, $x \in (0,1)$, $\epsilon > -1$, the researcher has

$$(1-\epsilon)^{2^k} = \left(1-\frac{1}{2^{1+\epsilon}}\right)^{2^k} < e^{-2^{k-(1+\epsilon)}}.$$

Therefore, for $k \geq [1+\epsilon]+1$ the researcher has $(1-\epsilon)^{2^k} < e^{-1}$ and so

$$S_2 = \lesssim \frac{1}{1+\epsilon} \sum_{k=[1+\epsilon]+1}^{1+\epsilon} 2^{k-(1+\epsilon)} \ e^{-2^{k-(1+\epsilon)}} \lesssim \frac{1}{1+\epsilon}.$$

For $k \leq [1+\epsilon]$ the researcher uses inequality

$$(1-\epsilon)^{-2^k} - (1+\epsilon) > e^{2^{k-(1+\epsilon)}} - 1 + \frac{1}{1+\epsilon} > 2^{k-(1+\epsilon)} + \frac{1}{1+\epsilon}.$$

Thus,

$$S_1 \lesssim \frac{1}{1+\epsilon}\sum_{k=1}^{[1+\epsilon]} 2^k \frac{\epsilon}{(1-\epsilon)^{-2k}-(1+\epsilon)} < \frac{1}{1+\epsilon}\sum_{k=1}^{[1+\epsilon]} \frac{2^{k-(1+\epsilon)}}{2^{k-(1+\epsilon)}+\frac{1}{1+\epsilon}} .$$

We split this sum into two more sums, over k such that $2^{k-(1+\epsilon)} < \frac{1}{1+\epsilon}$ and $2^{k-(1+\epsilon)} \geq \frac{1}{1+\epsilon}$. Then the researcher has

$$S_1 \lesssim \frac{1}{1+\epsilon} \sum_{1<k<(1+\epsilon)-\frac{\log(1+\epsilon)}{\log 2}} 2^{k-(1+\epsilon)}.(1+\epsilon) + \frac{1}{1+\epsilon} \sum_{(1+\epsilon)-\frac{\log(1+\epsilon)}{\log 2} \le k \le [1+\epsilon]} 2^{k-(1+\epsilon)}.2^{(1+\epsilon)} \lesssim 2^{-}\frac{\log(1+\epsilon)}{\log 2} + \frac{\log(1+\epsilon)}{(1+\epsilon)\log 2} \lesssim \frac{1}{1+\epsilon} + \frac{\log(1+\epsilon)}{(1+\epsilon)\log 2}.$$

Thus, the researcher has shown that

$$\|\mathcal{A}\|^*_{\mathcal{A}^0_{1,\infty}} \lesssim \frac{1}{1+\epsilon} + \frac{\log(1+\epsilon)}{(1+\epsilon)\log 2} \lesssim \frac{\log\log N}{\log N}.$$

3.1.2. Proof of Theorem 2

Applying Lemma 1 to $\mu = \mu^\star$ with $\left(1+\epsilon, \frac{1}{2}+3\epsilon\right) = (\infty, 1)$ the researcher obtains

$$\prod_{i=1}^{N} |\ltimes_i|.\, cap_{\mathcal{A}^0_{\infty,1}}(\mu^\star) \gtrsim \frac{1}{\|\mathcal{A}\|^*_{\mathcal{A}^0_{1,\infty}}},$$

because $\prod_{i=1}^{N} |\ltimes_i| = (1+\epsilon)^{1+\epsilon} < e^{-1}$. It remains to apply Proposition 1.

3.2 Upper bounds on $cap_{\mathcal{A}^0_{\left(1+\epsilon,\ \frac{1}{2}+3\epsilon\right)}}(\mu)$ for general values of $\left(1+\epsilon\,,\ \frac{1}{2}+3\epsilon\right) \in [1,\infty]^2$.

The researcher statements the constants in $\lesssim$ relations may depend on $1+\epsilon\,,\ \frac{1}{2}+3\epsilon$ but not on N.

Theorem 3. Given $\left(1+\epsilon\,,\ \frac{1}{2}+3\epsilon\right) \in [1,\infty]^2$ and $\mu = (\ltimes_1, \cdots\cdots, \ltimes_N) \in \mathbb{D}^N_*$ the following upper estimates on $\mathrm{cap}_{\mathcal{A}^0_{\left(1+\epsilon,\ \frac{1}{2}+3\epsilon\right)}}(\mu)$ hold depending on the region to which $\left(1+\epsilon\,,\ \frac{1}{2}+3\epsilon\ \right)$ belongs.

If $\left(1+\epsilon\,,\ \frac{1}{2}+3\epsilon\right) \in [1,\infty]^2$ then

$$\mathrm{cap}_{\mathcal{A}^0_{\left(1+\epsilon,\ \frac{1}{2}+3\epsilon\right)}}(\mu) \lesssim \frac{(\log N)^{\frac{1-3\epsilon}{2(1+3\epsilon)}}}{\prod_{i=1}^{N}|\ltimes_i|}.$$

If $1 < 1+\epsilon < \frac{1}{2}+3\epsilon \le \infty$ and $\epsilon \ge \frac{1}{2}$, then

$$\mathrm{cap}_{\mathcal{A}^0_{1+\epsilon,\frac{1}{2}+3\epsilon}}(\mu) \lesssim \frac{1}{\prod_{i=1}^{N}|\ltimes_i|}.$$

If $1 < 1+\epsilon < \frac{1}{2}+3\epsilon \le \infty$ and $\epsilon \ge 1$, then

$$\mathrm{cap}_{\mathcal{A}^0_{\left(1+\epsilon,\ \frac{1}{2}+3\epsilon\right)}}(\mu) \lesssim \frac{(\log N)^{\epsilon/\frac{1}{2}+\frac{7}{2}\epsilon+\epsilon^2}}{\prod_{i=1}^{N}|\ltimes_i|}.$$

3.1.3. Proof of theorem 3

The researcher considers the simplest function

$$f^2 = \frac{\mathcal{A}_\mu}{\mathcal{A}_\mu(0)} = (-1)^N \frac{\mathcal{A}_\mu}{\prod_{i=1}^{N}\ltimes_i}.$$

$$\mathrm{cap}_{\mathcal{A}^0_{1+\epsilon,\ \frac{1}{2}+3\epsilon}}(\mu) \leq \frac{\|\mathcal{A}_\mu\|_{\mathcal{A}^0_{1+\epsilon,\ \frac{1}{2}+3\epsilon}}}{\prod_{i=1}^{N}|\ltimes_i|},$$

And the statement follows from Proposition 10 in all cases except $\epsilon = -1$, where an application of Proposition 10 will lead to a substantially worse growth order. To treat the case $\epsilon = -1$ the researcher uses the test function from [10]. Put $\epsilon = \frac{1}{N}$ and consider the finite Blaschke product

$\tilde{\mathcal{A}}$ with zeros $(1-\epsilon)\ltimes_1, \cdots\cdots, (1-\epsilon)\ltimes_N$,

$$\tilde{\mathcal{A}}(z) = \prod_{i=1}^{N} \frac{z-(1-\epsilon)\ltimes_i}{1-(1-\epsilon)\bar{\ltimes}_i z}.$$

Let

$$f^2(z) = (-1)^N \frac{\tilde{\mathcal{A}}((1-\epsilon)(z))}{(1-\epsilon)^N \prod_{i=1}^{N}\ltimes_i}.$$

Clearly f^2 satisfies $f^2(0) = 1$ and $f^2(\ltimes_i) = 0$ for $i = 1, \cdots\cdots, (1+\epsilon)$. Since $(1-\epsilon)^N \asymp 1$, the researcher has

$$\prod_{i=1}^{N}|\ltimes_i| \cdot \left|f'\rho e^{i(1+\epsilon)}\right| \lesssim \left|\tilde{\mathcal{A}}'(1-\epsilon)\rho e^{i(1+\epsilon)})\right| \lesssim \frac{1}{1-(1-\epsilon)\rho}.$$

Then, for $\frac{1}{6} \leq \epsilon < \infty$,

$$\left(\prod_{i=1}^{N}|\ltimes_i|\right)^{\frac{1}{2}+3\epsilon} \cdot \|f^2\|^{\frac{1}{2}+3\epsilon}_{\mathcal{A}^0_{\infty,\frac{1}{2}+3\epsilon}} \lesssim \int_0^{1-\frac{1}{N}} \frac{d(1+\epsilon)}{\epsilon} + \int_{1-\frac{1}{N}}^{1} \frac{d(1+\epsilon)}{-\epsilon} \lesssim \log N.$$

Theorem 3 is proved.

Remark 1. The upper bound is attained by any sequence $\mu = (\ltimes_1, \cdots\cdots, \ltimes_N) \in \mathbb{D}_*^N$ such that $|\ltimes_i| \geq 1 - 1/N$ for all $i = 1, \cdots\cdots, N$.

3.2. Lower estimates on $\mathbf{cap}_{\mathcal{A}^0_{1+\epsilon,\ \frac{1}{2}+3\epsilon}}(\mu^\star)$

The researcher gives theorem and derives quantitive lower estimates on $cap_{\mathcal{A}^0_{1+\epsilon,\ \frac{1}{2}+3\epsilon}}(\mu^\star)$ for $1 \leq \frac{1}{2} + 3\epsilon \leq 1+\epsilon \leq \infty$.This proves, in particular, the sharpness of theorem 3 for $(1+\epsilon,\ \frac{1}{2}+3\epsilon)$ if $\epsilon > \infty$.

Theorem 4. Let $\mu^* \in \mathbb{D}_*^N$, and $\mathcal{A}^\star$ be defined by (12) and (13), and let $\left(1+\epsilon,\ \frac{1}{2}+3\epsilon\right) \in [1,\infty]^2$ be such that $1 \leq \frac{1}{2} + 3\epsilon \leq 1+\epsilon \leq \infty$. Then

$$\prod_{i=1}^{N}|\ltimes_i|.\,\mathrm{cap}_{\mathcal{A}^0_{1+\epsilon,\ \frac{1}{2}+3\epsilon}}(\mu^\star) \gtrsim (\log N)^{\frac{\frac{\epsilon}{1}+\frac{7}{2}\epsilon+\epsilon^2}{2}} \quad ,\epsilon < \infty \qquad \ldots.(16)$$

$$\prod_{i=1}^{N}|\ltimes_i|.\,\mathrm{cap}_{\mathcal{A}^0_{\infty,\ \frac{1}{2}+3\epsilon}}(\mu^\star) \gtrsim \frac{(logN)^{\frac{\frac{1}{1}+3\epsilon}{2}}}{logNlogN}. \qquad \ldots\ldots\ldots\ (17)$$

In particular, for $1+\epsilon$, $\frac{1}{2}+3\epsilon$ and $\epsilon < \infty$,

$$\prod_{i=1}^{N} |\ltimes_i|.\,\text{cap}_{\mathcal{A}^{0}_{1+\epsilon,\ \frac{1}{2}+3\epsilon}}(\mu^\star) \asymp (\log N)^{\frac{\frac{\epsilon}{1}+\frac{7}{2}\epsilon+\epsilon^2}{2}} \quad \dots\dots \quad (18)$$

However, for $1 \le 6\epsilon \le \epsilon \le 3$ there is still a certain gap between the upper and lower estimates for the capacities:

$$(\log N)^{\epsilon/\frac{1}{2}+\frac{7}{2}\epsilon+\epsilon^2} \lesssim \prod_{i=1}^{N} |\ltimes_i|\,.\,\text{cap}_{\mathcal{A}^{0}_{1+\epsilon,\ \frac{1}{2}+3\epsilon}}(\mu^\star) \lesssim (\log N)^{\frac{3}{2}-3\epsilon/1+6\epsilon}.$$

Let us consider the diagonal case $\frac{1}{6} \le \epsilon = \epsilon < 1$. Rudin [16] showed that there exists a Blaschke product that is not contained in $\mathcal{A}^{0}_{1,1}$, see also [12]. Vinogradov [21, Theorem 3.11] extended Rudin's result to $\mathcal{A}^{0}_{(1+\epsilon)(1+\epsilon)}$ for $-1 < \epsilon < 1$ see [23].These results perhaps suggest that the expression in the middle be unbounded for $\frac{1}{6} \le \epsilon = \epsilon < 1$. Indeed, unboundedness would follow if the researcher knows that there are Blaschke sequences that are not zero sets for $\mathcal{A}^{0}_{(1+\epsilon)(1+\epsilon).}$ However, the existence of such Blaschke sequences appears to be an open question. Results about zero sets for $\mathcal{A}^{0}_{(1+\epsilon)(1+\epsilon).}$ also for $\epsilon > 1$, can be found in [6].

Instead, the researcher gives a different, qualitative argument showing that, in case $\frac{1}{6} \le \epsilon = \epsilon < 1$.The expression in the middle may be unbounded.

Theorem 5. For each $N \in \mathbb{N}$ there exists a finite sequence $\mu_N \in \mathbb{D}^{N}_{*}$ such that for all $0 \le \epsilon < 1$, the researcher has

$$\lim_{N\to\infty} \prod_{\ltimes \in \mu_N} |\ltimes_i| \cdot cap_{\mathcal{A}^{0}_{(1+\epsilon),(1+\epsilon)}}(\mu_N) = \infty.$$

It will be convenient to extend the definition of $\text{cap}_{\mathcal{A}^{1+\epsilon}_{1+\epsilon,\ \frac{1}{2}+3\epsilon}}(\mu)$ to possibly infinite sequences μ in the obvious way. The infimum over the empty set is understood to be $+\infty$, so that $\text{cap}_{\mathcal{A}^{1+\epsilon}_{1+\epsilon,\ \frac{1}{2}+3\epsilon}}(\mu) = +\infty$ in case μ is a uniqueness set for $\text{cap}_{\mathcal{A}^{1+\epsilon}_{1+\epsilon,\ \frac{1}{2}+3\epsilon}}$. Our approach to bound $\text{cap}_{\mathcal{A}^{1+\epsilon}_{1+\epsilon,\ \frac{1}{2}+3\epsilon}}(\mu)$ from below is based on a duality method. Namely, the key step of the proof is the following lemma:

Proposition 2. If $1 \le 1+\epsilon \le \frac{1}{2} + 3\epsilon \le \infty$, then

$$\|\mathcal{A}^\star\|^{*}_{\mathcal{A}^{0}_{1,(\frac{1}{2}+3\epsilon)}} \lesssim \frac{1}{(\log N)^{-\frac{1}{2}+2\epsilon/(\frac{1}{2}+\frac{7}{2}\epsilon+3\epsilon^2)}}, \qquad \epsilon > 0,$$

$$\|\mathcal{A}^\star\|^{*}_{\mathcal{A}^{0}_{1,(\frac{1}{2}+3\epsilon)}} \lesssim \frac{\log N \log N}{(\log N)^{1-1/\frac{1}{2}+3\epsilon}}.$$

The idea of the proof of Theorem 5 is also to use duality. In case $\epsilon > 0$, the dual norm turns out to be the Bloch semi-norm. An obstacle to this strategy is a result of Baranov, Kayumov, and Nasyrov [4], according to which the Bloch semi-norm of finite Blascke products is bounded below by a universal constant. Instead, the researcher works with infinite Blasche prouducts, and carry out an approximation argument.

3.2.1. Proofs of Theorem 4 and Theorem 5

Proof of Proposition 2

As in the proof of Proposition 1, for simplicity the researcher writes $\mathcal{A}$ instead $\mathcal{A}^\star$ throughout the proof.

- Step 1: If $\epsilon = \infty$. Note that the case $\epsilon = 0$ is already covered by Proposition 1. starts with case $0 < \epsilon \le 1$. The researcher has to prove that

$$\sup_{1<\epsilon<0} \epsilon \left(\int_0^{2\pi} \left|\mathcal{A}'(1-\epsilon)e^{i(1+\epsilon)}\right|^{1+\epsilon} d(1+\epsilon)\right)^{1/(1+\epsilon)} \lesssim \frac{1}{(\log N)^{1/(1+\epsilon)}}. \qquad (19)$$

It follows from (15) that

$$I = \epsilon^{1+\epsilon} \int_0^{2\pi} \left|\mathcal{A}'(1-\epsilon)e^{i(1+\epsilon)}\right|^{1+\epsilon} d(1+\epsilon) \lesssim \int_0^{2\pi} \left(\frac{1}{1+\epsilon} \sum_{k=1}^{1+\epsilon} 2^k (1-\epsilon)^{2^k-1} \frac{\epsilon}{\left|1-(1+\epsilon)(1-\epsilon)^{2^k} e^{i2^k}\right|^2}\right)^{1+\epsilon} d(1+\epsilon).$$

Since for $-1 < \epsilon \le 1$ and any $(1+\epsilon)_k \ge 0$ one has

$$\left(\sum_k (1+\epsilon)_k\right)^{1+\epsilon} \le \left(\sum_k (1+\epsilon)_k^{1+\epsilon/2}\right)^2,$$

The researcher concludes that

$$I \le \frac{1}{(1+\epsilon)^{1+\epsilon}} \int_0^{2\pi} \left(\sum_{k=1}^{1+\epsilon} \frac{(2^k(1-\epsilon)^{2^k-1} - (\epsilon))^{1+\epsilon/2}}{\left|1-(1+\epsilon)(1-\epsilon)^{2^k} e^{i2^k}\right|^{1+\epsilon}}\right)^2 d(1+\epsilon)$$

$$\lesssim \frac{1}{(1+\epsilon)^{1+\epsilon}} \int_0^{2\pi} \sum_{k\le j\le(1+\epsilon)} \frac{\left(2^k(1-\epsilon)^{2^k-1} - (\epsilon)\right)^{1+\epsilon/2} \left(2^j(1-\epsilon)^{2^j-1} - (\epsilon)\right)^{1+\epsilon/2}}{\left|1-(1+\epsilon)(1-\epsilon)^{2^k} e^{i2^k(1+\epsilon)}\right|^{1+\epsilon} \left|1-(1+\epsilon)(1-\epsilon)^{2^{kj}} e^{i2^j(1+\epsilon)}\right|^{1+\epsilon}} d(1+\epsilon)$$

$$\lesssim \frac{1}{(1+\epsilon)^{1+\epsilon}} \int_0^{2\pi} \sum_{k\le j\le(1+\epsilon)} \frac{\left(2^k(1-\epsilon)^{2^k-1} - (\epsilon)\right)^{1+\epsilon/2} \left(2^j(1-\epsilon)^{2^j-1} - (\epsilon)\right)^{1+\epsilon/2}}{\left|1-(1+\epsilon)(1-\epsilon)^{2^k} e^{i2^k(1+\epsilon)}\right|^{1+\epsilon} ((1-(1+\epsilon)(1-\epsilon)^{2^j})^{1+\epsilon}} d(1+\epsilon).$$

After integration with respect to (1+ϵ) and using a well-known estimate of Forelli and Rudin (see [8, Theorem 1.7]) the researcher gets

$$I \lesssim \frac{1}{(1+\epsilon)^{1+\epsilon}} \sum_{k\le j\le(1+\epsilon)} \frac{\left(2^k(1-\epsilon)^{2^k-1} - (\epsilon)\right)^{1+\epsilon/2} \left(2^j(1-\epsilon)^{2^j-1} - (\epsilon)\right)^{1+\epsilon/2}}{\left((1-(1+\epsilon)(1-\epsilon)^{2^k}\right)^{\epsilon} \left((1-(1+\epsilon)(1-\epsilon)^{2^j}\right)^{1+\epsilon}}$$

$$\lesssim \frac{1}{(1+\epsilon)^{1+\epsilon}} \sum_{k\le j\le(1+\epsilon)} \frac{\left(2^k(1-\epsilon)^{2^k-1} - (\epsilon)\right)^{1+\epsilon/2} \left(2^j(1-\epsilon)^{2^j-1} - (\epsilon)\right)^{1+\epsilon/2}}{\left((1-(1+\epsilon)(1-\epsilon)^{2^k}\right)^{\epsilon/2} \left((1-(1+\epsilon)(1-\epsilon)^{2^j}\right)^{\epsilon/2}}$$

$$\lesssim \frac{1}{(1+\epsilon)^{1+\epsilon}} \left(\sum_{k=1}^{1+\epsilon} \frac{(2^k(1-\epsilon)^{2^k-1} - (\epsilon))^{1+\epsilon/2}}{\left(1-(1+\epsilon)(1-\epsilon)^{2^k}\right)^{\epsilon/2}}\right)^2.$$

Thus, the researcher shows that

$$S = \frac{1}{(1+\epsilon)^{1+\epsilon/2}} \sum_{K=1}^{1+\epsilon} \frac{(2^k(1-\epsilon)^{2^k-1} - (\epsilon))^{1+\epsilon/2}}{\left(1-(1+\epsilon)(1-\epsilon)^{2^k}\right)^{\epsilon/2}} \le \frac{1}{\sqrt{1+\epsilon}}.$$

If $\epsilon \le \frac{1}{2}$, then, clearly, $S \lesssim (1+\epsilon)^{-1+\epsilon/2} \le (1+\epsilon)^{-1/2}$. Now, let $\epsilon = 1/2^{1+\epsilon}$ where $\epsilon \ge 0$. If $k = [1+\epsilon]+1$. Then $(1-\epsilon)^{2^k} < e^{-2^{k-(1+\epsilon)}} \le 1/e$ and

$$\frac{1}{(1+\epsilon)^{\frac{1+\epsilon}{2}}}\sum_{k=[1+\epsilon]+1}^{1+\epsilon}\frac{\left(2^k(1-\epsilon)^{2^k-1}-(\epsilon)\right)^{1+\epsilon/2}}{\left(1-(1+\epsilon)(1-\epsilon)^{2^k}\right)^{\frac{\epsilon}{2}}}\lesssim\frac{1}{(1+\epsilon)^{\frac{1+\epsilon}{2}}}\sum_{k=[1+\epsilon]+1}^{1+\epsilon}\left(2^{k-(1+\epsilon)}e^{2^{k-(1+\epsilon)}}\right)^{\frac{\epsilon}{2}}\lesssim\frac{1}{(1+\epsilon)^{\frac{1+\epsilon}{2}}}.$$

Note that $(1-\epsilon)^{2^k}=\left((1-\frac{1}{2^{1+\epsilon}})\right)^{2^k}\geq(1-\frac{1}{2})^2$ for $k=[1+\epsilon]$ and, therefore, as in the proof of Proposition 1,

$$\left|1-(1+\epsilon)(1-\epsilon)^{2^k}\right|\gtrsim r^{-2^k}-(1+\epsilon)=(1-\epsilon)^{-2^k}-1+1/(1+\epsilon)\geq 2^{k-(1+\epsilon)}+1/(1+\epsilon).$$

As in the proof of Proposition 1 the researcher splits the sum into two parts. For $1\leq k<(1+\epsilon)-\frac{\log(1+\epsilon)}{\log 2}$ the researcher has $2^{k-(1+\epsilon)}<1/(1+\epsilon)$ and, therefore.

$$\frac{1}{(1+\epsilon)^{1+\epsilon/2}}\sum_{k<(1+\epsilon)-\log(1+\epsilon)/\log 2}\frac{\left(2^k(1-\epsilon)^{2^k-1}-(\epsilon)\right)^{1+\epsilon/2}}{\left(1-(1+\epsilon)(1-\epsilon)^{2^k}\right)^{1+2\epsilon/2}}$$
$$\lesssim\frac{1}{(1+\epsilon)^{1+\epsilon/2^{(1+\epsilon)^{(1+\epsilon)}-1/2}}}\sum_{k<(1+\epsilon)-\log(1+\epsilon)/\log 2}2^{(k-(1+\epsilon)(1+\epsilon)/2}$$
$$\lesssim(1+\epsilon)^{1+\epsilon/3/2}2^{(-\log n/\log 2)(1+\epsilon)/2}=(1+\epsilon)^{-\frac{1}{2}}.$$

Finally, for $(1+\epsilon)-\frac{\log(1+\epsilon)}{\log 2}\leq k\leq[1+\epsilon]$ the researcher has $2^{k-(1+\epsilon)}\geq 1/(1+\epsilon)$ and so

$$\frac{1}{(1+\epsilon)^{1+\epsilon/2}}\sum_{(1+\epsilon)-\log(1+\epsilon)/\log 2\leq k\leq[1+\epsilon]}\frac{\left(2^k(1-\epsilon)^{2^k-1}-(\epsilon)\right)^{1+\epsilon/2}}{\left(1-(1+\epsilon)(1-\epsilon)^{2^k}\right)^{1+2\epsilon/2}}$$
$$\lesssim\frac{1}{(1+\epsilon)^{1+\epsilon/2}}\sum_{(1+\epsilon)-\log(1+\epsilon)/\log 2\leq k\leq[1+\epsilon]}2^{(k-(1+\epsilon))(1+\epsilon)/2}\,2^{((1+\epsilon)-k)(1+2\epsilon)/2)}$$
$$\lesssim 1+\epsilon^{-(1+\epsilon)/2}\,2^{(\log(1+\epsilon)/\log 2))(1+2\epsilon)/}=(1+\epsilon)^{-1/2}.$$

Thus, the researcher shows that $S=(1+\epsilon)^{-1/2}$ for $0<\epsilon\leq 1$, and so

$$I=\epsilon^{1+\epsilon}\int_0^{2\pi}\left|\mathcal{A}'(1-\epsilon)e^{i(1+\epsilon)}\right|^{1+\epsilon}\,d(1+\epsilon)\lesssim S^2\lesssim\frac{1}{\log N}.$$

The estimate remains true for $\epsilon>1$ since by the Schwarz-Pick inequality, the researcher has

$$\left|\mathcal{A}'(1-\epsilon)e^{i(1+\epsilon)}\right|^{1+\epsilon}\leq(2\epsilon-\epsilon^2)^{1-\epsilon}\left|\mathcal{A}'(1-\epsilon)e^{i(1+\epsilon)}\right|^2.$$

Step 2: the case $1\leq 1+\epsilon\leq\frac{1}{2}+3\epsilon\leq\infty$. The researcher has to shows that

$$\int_0^1\epsilon^{-\frac{1}{2}+3\epsilon}\left(\int_0^{2\pi}\left|\mathcal{A}'(1-\epsilon)e^{i(1+\epsilon)}\right|^{1+\epsilon}\,d(1-\epsilon)\right)^{\frac{1}{2}+3\epsilon/1+\epsilon}\lesssim\frac{1}{(\log N)^{\frac{1}{2}+3\epsilon/\epsilon}},$$

(respectively $\lesssim\frac{(\log\log N)^{\frac{1}{2}+3\epsilon}}{(\log N)^{-\frac{1}{2}+3\epsilon}}$ in case $\epsilon>0$). It follows from (19) that for $0<\epsilon<\infty$

$$\int_0^{1-1/N^2} \epsilon^{-\frac{1}{2}+3\epsilon} \left(\int_0^{2\pi} \left|\mathcal{A}'(1-\epsilon)e^{i(1+\epsilon)}\right|^{1+\epsilon} d(1+\epsilon)\right)^{\frac{1}{2}+3\epsilon/1+\epsilon} d(1-\epsilon) \lesssim \frac{1}{(\log N)^{\frac{1}{2}+3\epsilon/1+\epsilon}} \int_0^{1-1/N^2} \frac{d(1-\epsilon)}{\epsilon} \lesssim \frac{1}{(\log N)^{\frac{1}{2}+3\epsilon/1+\epsilon}},$$

While for $\epsilon = 0$ the researcher has by Proposition 1 that

$$\int_0^{1-1/N^2} \epsilon^{-\frac{1}{2}+3\epsilon} \left(\int_0^{2\pi} \left|\mathcal{A}'(1-\epsilon)e^{i(1+\epsilon)}\right|^{1+\epsilon} d(1+\epsilon)\right)^{\frac{1}{2}+3\epsilon/1+\epsilon} d(1-\epsilon) \lesssim \frac{(\log\log N)^{\frac{1}{2}+3\epsilon}}{(\log N)^{-\frac{1}{2}+3\epsilon}}.$$

The researcher notes that, since

$$\int_0^{2\pi} \left|\mathcal{A}'(1-\epsilon)e^{i(1+\epsilon)}\right|^{1+\epsilon} d(1+\epsilon) \leq \int_0^{2\pi} \left|\mathcal{A}'(1-\epsilon)e^{i(1+\epsilon)}\right|^{1+\epsilon} d(1+\epsilon) = 2\pi N,$$

the researcher has by the Schwarz-Pick inequality

$$\int_0^{2\pi} \left|\mathcal{A}'(1-\epsilon)e^{i(1+\epsilon)}\right|^{1+\epsilon} d(1+\epsilon) \leq \frac{1}{(2\epsilon-\epsilon^2)^{\epsilon}} \int_0^{2\pi} \left|\mathcal{A}'(1-\epsilon)e^{i(1+\epsilon)}\right|^{1+\epsilon} d(1+\epsilon) \leq \frac{2\pi N}{(2\epsilon-\epsilon^2)^{\epsilon}}.$$

Therefore.

$$\int_{1-1/N^2}^{1} \epsilon^{-\frac{1}{2}+3\epsilon} \left(\int_0^{2\pi} \left|\mathcal{A}'(1-\epsilon)e^{i(1+\epsilon)}\right|^{1+\epsilon} d(1+\epsilon)\right)^{\frac{1}{2}+3\epsilon/1+\epsilon} d(1-\epsilon)$$
$$\lesssim N^{\frac{1}{2}+3\epsilon/1+\epsilon} \int_{1-\frac{1}{N^2}}^{1} \epsilon^{-\frac{1}{2}+\frac{5}{2}\epsilon-3\epsilon^2/\epsilon} d(1-\epsilon) = N^{\frac{1}{2}+3\epsilon/\epsilon} \int_{1-\frac{1}{N^2}}^{1} \epsilon^{\frac{1}{2}+3\epsilon/\epsilon} \lesssim N^{-\frac{1}{2}+3\epsilon/\epsilon} = O\left(\frac{1}{(\log N)^{\frac{1}{2}+3\epsilon/\epsilon}}\right).$$

Combining the above estimates the researcher comes to the conclusion of the Proposition.

Lemma 1. let $0 \leq \epsilon$, $\epsilon \leq \infty$ and a finite sequence μ in $\mathbb{D}$, the researcher has

$$\prod_{\ltimes\in\mu} |\ltimes| \cdot cap_{\mathcal{A}^0_{1+\epsilon,\frac{1}{2}+3\epsilon}}(\mu) \gtrsim \frac{1-\prod_{\ltimes\in\mu}|\ltimes|^2}{\left\|\mathcal{A}_\mu\right\|^*_{\mathcal{A}^0_{(1+\epsilon)',\left(\frac{1}{2}+3\epsilon\right)'}}},$$

where $\mathcal{A}_\mu$ is the Blaschke product with the zero set μ and $(1+\epsilon)'$, $\left(\frac{1}{2}+3\epsilon\right)'$ are the exponents conjugate to $1+\epsilon$, $\frac{1}{2}+3\epsilon$. The same estimate is true for arbitrary Blaschke sequence μ in $\mathbb{D}_*$ in case $0 \leq \epsilon$, $\epsilon \leq \frac{1}{2}$.

To show the lower estimate (15) it remains to apply Lemma 1 to $\mu = \mu^\star$ and estimate from above the Besov semi norm of $\mathcal{A}^\star$. Namely the researcher proves the following.

3.2.2. Proof of Lemma 1

Suppose first that μ is a finite sequence in $\mathbb{D}_*$, say $|\mu| = N$. Let f^2 be a function that is analytic in a neighborhood of $\overline{\mathbb{D}}$ such that $f^2(0) = 1$ and $f^2|\mu = 0$. Then the researcher has (writing $\mathcal{A} = \mathcal{A}_\mu$)

$$\langle f^2, \mathcal{A}\rangle = \frac{f^2(0)}{\mathcal{A}(0)} = \frac{1}{\prod_{\ltimes\in\mu}\ltimes}.$$

The researcher shows that

$$\langle f^2, \mathcal{A}\rangle \leq |f^2(0)||\mathcal{A}(0)| + C\|f^2\|^*_{\mathcal{A}^0_{(1+\epsilon)',\left(\frac{1}{2}+3\epsilon\right)'}} = \prod_{\propto\in\mu}|\propto| + C\|f^2\|^*_{\mathcal{A}^0_{1+\epsilon,\frac{1}{2}+3\epsilon}}\|\mathcal{A}\|^*_{\mathcal{A}^0_{(1+\epsilon)',\left(\frac{1}{2}+3\epsilon\right)'}}.$$

Thus,

$$\prod_{\propto\in\mu}|\propto|\cdot\|f^2\|^*_{\mathcal{A}^0_{1+\epsilon,\frac{1}{2}+3\epsilon}} \geq \frac{1-\prod_{\propto\in\mu}|\propto|^2}{C\|\mathcal{A}\|^*_{\mathcal{A}^0_{(1+\epsilon)',\left(\frac{1}{2}+3\epsilon\right)'}}}. \quad \ldots\ldots (20)$$

Now, let $f^2 \in \mathcal{A}^0_{1+\epsilon,\frac{1}{2}+3\epsilon}$ be an arbitrary function such that $f^2(0) = 1$ and $f^2|\mu = 0$. Let $1 < \epsilon < 0$ be such that $\frac{1}{1-\epsilon} \subset \mathbb{D}$. Then $f^2_{(1-\epsilon)}$ vanishes on $\frac{1}{1-\epsilon}\mu$, hence by what has already been proved,

$$(1-\epsilon)^N\prod_{\propto\in\mu}|\propto|\cdot\left\|f^2_{(1-\epsilon)}\right\|^*_{\mathcal{A}^0_{1+\epsilon,\frac{1}{2}+3\epsilon}} \geq \frac{1-(1-\epsilon)^{2N}\prod_{\propto\in\mu}|\propto|^2}{C\left\|\mathcal{A}_{\frac{1}{1-\epsilon}\mu}\right\|^*_{\mathcal{A}^0_{(1+\epsilon)',\left(\frac{1}{2}+3\epsilon\right)'}}}.$$

Recall that $\left\|f^2_{(1-\epsilon)}\right\|^*_{\mathcal{A}^0_{1+\epsilon,\frac{1}{2}+3\epsilon}} \leq \|f^2\|^*_{\mathcal{A}^0_{1+\epsilon,\frac{1}{2}+3\epsilon}}$. Moreover, $\mathcal{A}_{\frac{1}{1-\epsilon}\mu}$ converges to $\mathcal{A}_\mu$ uniformly in a neighborhood of $\mathbb{D}$ as $\epsilon \to 0$. So taking the limit $\epsilon \to 0$, the researcher concludes that (20) holds for arbitrary $f^2 \in \mathcal{A}^0_{1+\epsilon,\frac{1}{2}+3\epsilon}$ satisfying $f^2(0) = 1$ and $f^2|\mu = 0$. Taking the infimum over all admissible functions f^2, we obtain the lemma for finite sequences.

Let now $0 \leq \epsilon = \epsilon \leq \frac{1}{2}$ and let μ be a possibly infinite Blaschke sequence. Let $\mathcal{A} = \mathcal{A}_\mu$ and let $f^2 = \mathcal{A}^0_{(1+\epsilon),(1+\epsilon)}$ be function vanishing on μ with $f^2(0) = 1$. The researcher apply Lemma 8 to the functions $f^2_{(1-\epsilon)}$ and $\mathcal{A}_{(1-\epsilon)}$ to obtain the bound

$$\langle f^2_{(1-\epsilon)}, \mathcal{A}_{(1-\epsilon)}\rangle \leq |\mathcal{A}(0)| + C\|f^2\|^*_{\mathcal{A}^0_{1+\epsilon,1+\epsilon}}\|\mathcal{A}\|^*_{\mathcal{A}^0_{\frac{1}{2}+3\epsilon,\frac{1}{2}+3\epsilon}}$$

For all $\epsilon < 0$.

The classical Littlewood-Palcy inequality shows that $\mathcal{A}^0_{1+\epsilon,1+\epsilon} \subset H^{1+\epsilon} \subset H^1$ (see [9, Theorem 6], so $f^2_{(1-\epsilon)} \to f^2$ in the norm of H^1. Moreover, $\mathcal{A} \in H^\infty$ and $\mathcal{A}_{(1-\epsilon)} \to \mathcal{A}$ weak-∗ in H^∞. From this, it follows that

$$\langle f^2, \mathcal{A}\rangle - \langle f^2_{(1-\epsilon)}, \mathcal{A}_{(1-\epsilon)}\rangle = \langle f^2, \mathcal{A}\rangle + \langle f^2 - f^2_{(1-\epsilon)}, \mathcal{A}_{(1-\epsilon)}\rangle \to 0$$

as $\epsilon \to 0$. Thus,

$$\langle f^2, \mathcal{A}\rangle \leq |\mathcal{A}(0)| + C\|f^2\|^*_{\mathcal{A}^0_{1+\epsilon,1+\epsilon}}\|\mathcal{A}\|^*_{\mathcal{A}^0_{\left(\frac{1}{2}+3\epsilon\right)',\left(\frac{1}{2}+3\epsilon\right)'}}.$$

Using that $f^2 \in H^1$ vanishes on μ, the researcher mays factor $f^2 = \mathcal{A}g^2$ for the some $g^2 \in H^1$. Then

$$\langle f^2, \mathcal{A}\rangle = \langle f^2, 1\rangle = g^2(0) = \frac{1}{\mathcal{A}(0)}.$$

Combining the last two formulas and taking the infimum over all admissible $f^2 \in \mathcal{A}^0_{(1+\epsilon),(1+\epsilon)}$ again yields the desired inequality.

3.2.3. Lemma 2

Let $0 \leq \epsilon$, $\epsilon \leq \infty$. There exists a constant $C \geq 0$ such that for all functions f^2 and g^2 that are analytic in a neighborhood of $\overline{\mathbb{D}}$, the researcher has

$$\langle f^2, g^2\rangle \leq |f^2(0)||g^2(0)| + C\|f^2\|^*_{\mathcal{A}^0_{1+\epsilon,\frac{1}{2}+3\epsilon}} \|g^2\|^*_{\mathcal{A}^0_{(1+\epsilon)',\left(\frac{1}{2}+3\epsilon\right)'}},$$

where $(1+\epsilon)', \left(\frac{1}{2}+3\epsilon\right)'$ are the exponents conjugate to $1+\epsilon$, $\frac{1}{2}+3\epsilon$

3.2.4. proof of lemma 2

Denote by (h^2, g^2) the scalar product on the Bergman space A^2 defined by

$$(h^2, g^2) = \int_{\mathbb{D}} h^2(u)\overline{g^2(u)}\ dA(u), \qquad h^2, g^2 \in A^2,$$

where $dA(u) = \frac{dxdy}{\pi}$ is the normalized planar Lebesgue measure on $\mathbb{D}$. the researcher recalls the simplest form of Creen's formula,

$$\langle \varphi_1 + \varphi_2, \psi\rangle = ((\varphi_1 + \varphi_2)', S^*\psi) + ((\varphi_1 + \varphi_2(0)), \overline{\psi(0)}), \quad \ldots \ (21)$$

where S^* is the backward shift operator $S^*f^2 = f^2 - f^2(0))/z$ and φ_1, ψ are functions that are analytic in a neighborhood of $\overline{\mathbb{D}}$. The researcher wills also need to use the following integral formula. Recall that the fractional differentiation operator $D_{\epsilon-1,}$ $0< \epsilon < \infty$, is defined by $D_j(z^j) = \frac{\Gamma(j+1+\epsilon)}{(j+1)!\Gamma(1+\epsilon)} z^j$, j = 0, 1,2,….., and extends linearly and continuously to the whole space $\mathcal{H}ol(\mathbb{D})$. Then, for functions f^2, g^2 analytic in a neighborhood of $\overline{\mathbb{D}}$ and $0< \epsilon < \infty$, the researcher has

$$\int_{\mathbb{D}} f^2(u)\overline{g^2(u)}\ dA(u) = \epsilon \int_{\mathbb{D}} D_{\epsilon-1}\ f^2(u)\overline{g^2(u)}\ (1-|u|)^{\epsilon-1}\, dA(u) \ldots\ldots \qquad (22)$$

see[8, Lemma 1.20].

Let f^2, g^2 analytic in a neighborhood of $\overline{\mathbb{D}}$. Applying (21) the researcher gets

$$\langle f^2, g^2\rangle = (f^{2\prime}, S^*g^2) + f^2(0)g^2(0).$$

Then the researcher apply (22) to $\overline{(f^{2\prime}, S^*g^2)} = (S^*g^2, f^{2\prime})$ with $\epsilon = 2$:

$$\left(S^*g^2, f^{2'}\right) = 2\int_{\mathbb{D}} D_1(S^*g^2)(u)\overline{f^{2\prime}(u)}\,(1-|u|)^{\epsilon-1}\, dA(u)$$

$$= 2\int_0^1 (-2\epsilon - 3\epsilon^2 - \epsilon^3)\left(\int_{\mathbb{T}} D_1(S^*g^2)(1+\epsilon)(z)\,\overline{f^{2\prime}(1+\epsilon)(z)}\, d(1+\epsilon)(z)\right) d(1+\epsilon).$$

By Hölder's inequality

$$\left|\int_{\mathbb{T}} D_1(S^*g^2)(1+\epsilon)(z)\,\overline{f^{2\prime}(1+\epsilon)(z)}\, d(1+\epsilon)(z)\right| \leq \|f^{2\prime}_{1+\epsilon}\|_{L(1+\epsilon)}\|(D_1(S^*g^2)_{1+\epsilon}\|_{L(1+\epsilon)'}.$$

Since $D_1(S^*g^2) = \frac{1}{2}(S^*g^2 + g^{2\prime})$ and $(S^*g^2)(z) = \frac{1}{z}\int_0^1 (1+\epsilon)g^{2\prime}((1+\epsilon)(z))d(1+\epsilon)$, it follows that $\|D_1(S^*g^2)\|_{L(1+\epsilon)'} \lesssim \|g^{2\prime}\|_{L(1+\epsilon)'}$. The preceding estimates therefore give

$$|S^*g^2, f^{2\prime}| \lesssim \int_0^1 -\epsilon\left\|f^{2\prime}_{1+\epsilon}\right\|_{L(1+\epsilon)'}\left\|g^{2\prime}_{1+\epsilon}\right\|_{L(1+\epsilon)'} d(1+\epsilon).$$

Then (again by Hölder's inequality) the researcher gets

$$|S^* g^2, f^{2\prime}| \lesssim \|f^2\|^*_{\mathcal{A}^0_{(1+\epsilon),(\frac{1}{2}+3\epsilon)}} \|g^2\|^*_{(1+\epsilon)',(\frac{1}{2}+3\epsilon)'} \ ,$$

Lemma 3. Let $0 \leq \epsilon , \epsilon \leq \infty$. Let $\mu \subset \mathbb{D}\backslash\{0\}$ be an infinite sequence. For $N \in \mathbb{N}$, let μ_N consist of the first N points of μ. Then

$$\operatorname{cap}_{\mathcal{A}^0_{(1+\epsilon),\left(\frac{1}{2}+3\epsilon\right)}}(\mu) = \lim_{\epsilon\to\infty} \operatorname{cap}_{\mathcal{A}^0_{(1+\epsilon),\left(\frac{1}{2}+3\epsilon\right)}}(\mu_N).$$

As desired.

3.2.5. Proof of lemma 3

For simplicity, the researcher has abbreviate cap $= \operatorname{cap}_{\mathcal{A}^0_{(1+\epsilon),\left(\frac{1}{2}+3\epsilon\right)}}$. The inequality $\operatorname{cap}(\mu_N) = \operatorname{cap}(\mu)$ is trivial, so it suffices to show that $\operatorname{cap}(\mu) = \leq \liminf_{N\to\infty} \operatorname{cap}(\mu_N)$. Clearly, the researcher mays assume that the limit inferior is finite.

If $c > \liminf_{N\to\infty} \operatorname{cap}(\mu_N)$, then by definition of capacity, there exist a sequence (N_k) tending to infinity and functions $f^2_{N_k} \in \mathcal{A}^0_{(1+\epsilon),\left(\frac{1}{2}+3\epsilon\right)}$ such that $f^2_{N_k}$ vanishes on μ_{N_k}, $f^2_{N_k}(0) = 1$ and $\|f^2\|_{\mathcal{A}^0_{(1+\epsilon),(\frac{1}{2}+3\epsilon)}} \leq c$ for all k. Then $\left(f^2_{N_k}\right)_k$ is a normal family, so a subsequence converges locally uniformly on $\mathbb{D}$ to a holomorphic function f^2. By Fatou's lemma, $f^2 \in \mathcal{A}^0_{(1+\epsilon),\left(\frac{1}{2}+3\epsilon\right)}$ with $\|f^2\|_{\mathcal{A}^0_{(1+\epsilon),(\frac{1}{2}+3\epsilon)}} \leq c$, and f^2 vanishes on μ and $f^2(0) = 1$. Thus, $\operatorname{cap}(\mu) \leq c$.

Lemma 4. Let $\mathcal{A}$ be a Blaschke product of degree N. Then

$$\int_0^1 (1-\rho)^\epsilon \int_0^{2\pi} \left|\mathcal{A}'\rho e^{i(1+\epsilon)}\right|^{1+\epsilon} d(1+\epsilon)d\rho \lesssim \begin{cases} (logN)^{-\frac{\epsilon}{2}} \quad , & 0 \leq \epsilon \leq 1, \\ 1, & 0 \leq \epsilon < \infty, \end{cases} \quad \ldots\ldots\ldots (23)$$

and, for $\rho \in [0,1[$ and $0 \leq \epsilon < \infty$,

$$\int_0^{2\pi} \left|\mathcal{A}'(\rho e^{i(1+\epsilon)})\right|^{1+\epsilon} d(1+\epsilon) \lesssim \frac{N}{(1-\rho)^\epsilon} \ . \quad \ldots\ldots (24)$$

Proof of lemma 4. Since $\int_0^{2\pi} \left|\mathcal{A}'(\rho e^{i(1+\epsilon)})\right|^{1+\epsilon} d(1+\epsilon) \leq 2\pi N,\ \rho \in [0\ ,1]$, the researcher concludes that

$$\int_{1-1/N}^{1} \int_0^{2\pi} \left|\mathcal{A}'(\rho e^{i(1+\epsilon)})\right| d(1+\epsilon)d\rho \lesssim 1.$$

Note that

$$\int_0^1 \rho(1-\rho^2) \int_0^{2\pi} \left|f^{2\prime}(\rho e^{i(1+\epsilon)}\right|^2 d(1+\epsilon)d\rho = \pi \sum_{\epsilon=0}^{\infty} \frac{\epsilon+1}{\epsilon+2} |a_{\epsilon+1}|^2 \leq \pi\|f^2\|^2_{H^2} \leq \pi\|f^2\|^2_{H^\infty} \quad (25)$$

For any function $f(z) = \sum_{\epsilon\geq -1} a_{(\epsilon+1)} z^{\epsilon+1}$ in the hardy space H^2. Therefore.

$$\int_0^{1-1/N} \int_0^{2\pi} \left|\mathcal{A}'(\rho e^{i(1+\epsilon)})\right| d(1+\epsilon)d\rho$$

$$\lesssim \left(\int_0^{1-1/N} \int_0^{2\pi} (1-\rho)\left|\mathcal{A}'(\rho e^{i(1+\epsilon)})\right|^2 d(1+\epsilon)d\rho \right)^{1/2} \left(\int_0^{1-1/N} \int_0^{2\pi} \frac{d(1+\epsilon)d\rho}{1-\rho} \right)^{1/2} \lesssim (\log N)^{1/2} .$$

Thus, the inequality is already proved for $\epsilon = 1$ (simply apply (25) to B) and $\epsilon = 0$. For

$0 < \epsilon < 1$ inequality (23) follows from the Hölder's inequality with exponents $(-\epsilon)^{-1}$ and $(1-\epsilon)^{-1}$ (note that $\epsilon = \epsilon$) and the estimates for exponents 1 and 2. Finally, for $\epsilon > 1$ it follows from the estimate $(1-|z|)^2\,|\mathcal{A}'(z)| \le 1, z \in \mathbb{D}$, that $(1-\rho)^{\epsilon}\|\mathcal{A}'_{\rho}\|^{1+\epsilon}_{L^{1+\epsilon}(\mathbb{T})} \le (1-\rho)\|\mathcal{A}'_{\rho}\|^{1+\epsilon}_{L^{2}(\mathbb{T})}$ and the researcher can again apply (25).

The estimate (24) is obvious:

$$\int_0^{2\pi} \left|\mathcal{A}'(\rho e^{i(1+\epsilon)})\right|^{1+\epsilon} d(1+\epsilon) \lesssim \frac{1}{(1-\rho)^{\epsilon}} \int_0^{2\pi} \left|\mathcal{A}'\left(\rho e^{i(1+\epsilon)}\right)\right| d(1+\epsilon) \lesssim \frac{N}{(1-\rho)^{\epsilon}}.$$

Proposition 3. Let $(1+\epsilon), (\frac{1}{2}+3\epsilon) \in [1,\infty]^2, \mu = (\ltimes_1, \cdots\cdots, \ltimes_N) \in \mathbb{D}^N$ and $\mathcal{A} = \mathcal{A}_{\mu}$.

$$\text{If } (1+\epsilon), \left(\frac{1}{2}+3\epsilon\right) \in [1\,,2]^2, then$$

$$\|\mathcal{A}\|_{\mathcal{A}^0_{(1+\epsilon),(\frac{1}{2}+3\epsilon)}} \lesssim ((logN))^{1/\left(-\frac{1}{2}+3\epsilon/2\right)}.$$

$$0 \le \epsilon \le -\frac{1}{2}+3\epsilon \le \infty \text{ and} \ge \frac{1}{2}\text{, then}$$

$$\|\mathcal{A}\|_{\mathcal{A}^0_{(1+\epsilon),(\frac{1}{2}+3\epsilon)}} \lesssim 1\,.$$

$0 \le \epsilon \le -\frac{1}{2}+3\epsilon < \infty$ and $\ge \frac{1}{2}$, then

$$\|\mathcal{A}\|_{\mathcal{A}^0_{(1+\epsilon),(\frac{1}{2}+3\epsilon)}} \lesssim (\log N)^{\frac{1}{2}-2\epsilon/(1+\epsilon),(\frac{1}{2}+3\epsilon)}$$

$-\frac{1}{6} \le \epsilon < \infty$ and $\epsilon = -1$, then

$$\|\mathcal{A}\|_{\mathcal{A}^0_{(1+\epsilon),(\frac{1}{2}+3\epsilon)}} \lesssim N^{1/\frac{1}{2}+3\epsilon}.$$

The constants in the relations $\lesssim$ depend only on $(1+\epsilon), \left(\frac{1}{2}+3\epsilon\right)$, but not on N and μ. Moreover, in all inequalities the dependence on the growth on N is best possible.

Note that there is an essential difference between the case $\epsilon < \infty$ and $\epsilon = \infty$, where the growth is much faster.

3.2.6. Proof of Proposition 3

Let $0 \le \epsilon < \infty$. Then it follows from (23) that

$$\left(\|\mathcal{A}\|^*_{\mathcal{A}^0_{(1+\epsilon),(\frac{1}{2}+3\epsilon)}}\right)^{1+\epsilon} = \int_0^1 (1-\rho)^{\epsilon} \int_0^{2\pi} \left|\mathcal{A}'\rho e^{i(1+\epsilon)}\right|^{1+\epsilon} d(1+\epsilon)d\rho$$

$$\lesssim \begin{cases} (logN)^{-\frac{\epsilon}{2}} \quad , & 0 \le \epsilon \le 1, \\ 1, & \epsilon \ge 1. \end{cases}$$

Also trivially $\|\mathcal{A}\|_{\mathcal{A}^0_{\infty,\infty}} \lesssim 1$.

Now let $0 \le -\frac{1}{2}+3\epsilon \le \epsilon \le \infty$, the researcher writes

$$\frac{1}{2\pi}\left(\|\mathcal{A}\|^{*}_{\mathcal{A}^{0}_{(1+\epsilon),(\frac{1}{2}+3\epsilon)}}\right)^{\frac{1}{2}+3\epsilon} = \int_0^{1-1/N}(1-\rho)^{-\frac{1}{2}+3\epsilon}\left(\int_0^{2\pi}\left|\mathcal{A}'\rho e^{i(1+\epsilon)}\right|^{1+\epsilon} d(1+\epsilon)\right)^{\frac{1}{2}+3\epsilon/1+\epsilon} d\rho + \int_{1-1/N}^{1} (1-\rho)^{-\frac{1}{2}+3\epsilon}\left(\int_0^{2\pi}\left|\mathcal{A}'\rho e^{i(1+\epsilon)}\right|^{1+\epsilon} d(1+\epsilon)\right)^{\frac{1}{2}+3\epsilon/1+\epsilon} d\rho$$

$$= I_1 + I_2 .$$

The researcher shows that $I_2 \lesssim 1$. Indeed, applying (23) the researcher gets

$$I_2 \lesssim N^{\frac{1}{2}+3\epsilon/1+\epsilon}\int_{1-1/N}^{1}(1-\rho)^{\frac{1}{2}+2\epsilon/1+\epsilon}\,d\rho = N^{\frac{1}{2}+3\epsilon/1+\epsilon}\int_{1-1/N}^{1}(1-\rho)^{\frac{1}{2}+2\epsilon/1+\epsilon}\,d\rho \lesssim 1.$$

To estimate I_1 , the researcher apply the Hölder's inequality with exponents $1+\epsilon/\frac{1}{2}+3\epsilon$ and $1+\epsilon/\frac{1}{2}-2\epsilon$ to get (with an obvious modification for $\epsilon=\frac{1}{4}$)

$$I_1 \le \left(\int_0^{1-1/N}(1-\rho)^{\epsilon}\int_0^{2\pi}\int_0^{2\pi}\left|\mathcal{A}'\rho e^{i(1+\epsilon)}\right|^{1+\epsilon}\ d(1+\epsilon)d\rho\right)^{\frac{1}{2}+3\epsilon/1+\epsilon}\left(\int_0^{1-1/N}\frac{d\rho}{1-\rho}\right)^{\frac{1}{2}-2\epsilon/1+\epsilon}.$$

Hence, for $0<\epsilon\le 1$,

$I_1 \lesssim (logN)^{-\frac{3}{2}-3\epsilon/2} = (logN)^{\frac{3}{2}-3\epsilon/2}$, while for $\epsilon > 1$

$I_1 \lesssim (logN)^{\frac{1}{2}-3\epsilon/1+\epsilon}$.

Thus, the researcher has proved 3) and 1) for the case $\epsilon \ge \frac{1}{4}$.

If $1 \le 1+\epsilon < \frac{1}{2}+3\epsilon < \infty$ the researcher simply has

$$\|\mathcal{A}\|^{*}_{\mathcal{A}^{0}_{(1+\epsilon),(\frac{1}{2}+3\epsilon)}} = \left(\int_0^1 (1-\rho)^{-\frac{1}{2}+3\epsilon}\ \left\|\mathcal{A}'_\rho\right\|^{\frac{1}{2}+3\epsilon}_{L^{1+\epsilon}(\mathbb{T})}\,d\rho\right)^{1/\frac{1}{2}+3\epsilon}$$

$$\lesssim \left(\int_0^1 (1-\rho)^{-\frac{1}{2}+3\epsilon}\ \left\|\mathcal{A}'_\rho\right\|^{\frac{1}{2}+3\epsilon}_{L^{\frac{1}{2}+3\epsilon}(\mathbb{T})}\,d\rho\right)^{1/\frac{1}{2}+3\epsilon} \lesssim \begin{cases}(\log N)^{\frac{3}{2}-3\epsilon/1+6\epsilon} & \frac{1}{6}\le\epsilon\le\frac{1}{2}\,, \\ 1, & \epsilon\ge\frac{1}{2}\,. \end{cases}$$

The case $\epsilon=\infty$ is trivial by the Schwarz-Pick lemma. The proof of the statements 1)-3) is completed.

Consider the case $\epsilon=\infty$. If $\mathcal{A}(z)=\prod_{j=1}^{N}\frac{z-\ltimes_j}{1-\bar{\ltimes}_j z}$, then

$$\mathcal{A}'(z) = \sum_{j=1}^{N}\hat{\mathcal{A}}_j(z)\,\frac{1-\left|\ltimes_j\right|^2}{\left(1-\bar{\ltimes}_j\,z\right)^2},$$

Where $\hat{\mathcal{A}}_j(z)=\prod_{k\ne j}\frac{z-\ltimes_k}{1-\bar{\ltimes}_j z}$. Hence,

$$\left\|\mathcal{A}'_\rho\right\|_\infty \lesssim \sum_{j=1}^{N}\frac{1-\left|\ltimes_j\right|^2}{\left(1-\left|\ltimes_j\right|\rho\right)^2}$$

and, using again the fact that $\|\mathcal{A}'_\rho\|_\infty \leq (1-\rho)^{-1}$, the researcher gets

$$\left(\|\mathcal{A}\|^*_{\mathcal{A}^0_{\infty,\frac{1}{2}+3\epsilon}}\right)^{\frac{1}{2}+3\epsilon} = \int_0^1 (1-\rho)^{-\frac{1}{2}+3\epsilon} \ \|\mathcal{A}'_\rho\|_\infty^{\frac{1}{2}+3\epsilon} d\rho \ \leq \int_0^1 \|\mathcal{A}'_\rho\|_\infty \lesssim N.$$

Let us show that all estimates are sharp. The growth $(\log N)^{\frac{1}{2}-2\epsilon/\frac{1}{2}+\frac{7}{2}\epsilon+3\epsilon^2}$ in the case 3) is achieved by the product $\mathcal{A}^\star$ defined by (13). Indeed, for $1 \leq \frac{1}{2}+3\epsilon \leq 1+\epsilon < \infty$,

$$\|\mathcal{A}^\star\|^*_{\mathcal{A}^0_{1+\epsilon,\frac{1}{2}+3\epsilon}} \geq \prod_{i=1}^{N} |\ltimes_i| \cdot \text{cap}_{\mathcal{A}^0_{1+\epsilon,\frac{1}{2}+3\epsilon}}(\mu^\star) \gtrsim \ (\log N)^{\frac{1}{2}-2\epsilon/\frac{1}{2}+\frac{7}{2}\epsilon+3\epsilon^2}$$

By Theorem 4. In the case 2) the optimality of the estimate can be already seen on $\mathcal{A}(z) = z^N$.

For the case $0 \leq \epsilon$, $\epsilon \geq \frac{1}{2}$ one can use an example of a Blaschke product constructed in [3]: there exists a Blaschke product of order N such that

$$\int_0^{1-1/N} \int_0^{2\pi} \left|\mathcal{A}'(\rho e^{i(1+\epsilon)})\right| d(1+\epsilon) d\rho \ \geq c\sqrt{\log N}\,,$$

where $c > 0$ is an absolute constant; see the end in [3]. This construction is based on deep results of R. Bañuelos and C.N. Moore [2] related to Makarov's law of the iterated logarithm. An casy application of the Hölder's inequality shows that for $0 \leq \epsilon$, $\epsilon \geq \frac{1}{2}$

$$\int_0^{1-1/N} (1-\rho)^{-\frac{1}{2}+3\epsilon} \ \|\mathcal{A}'_\rho\|_{L^{1+\epsilon}(\mathbb{T})}^{\frac{1}{2}+3\epsilon} d\rho \gtrsim (\log N)^{\frac{3}{2}-3\epsilon/2}.$$

Finally, let us show that the estimate in the case 4) also is best possible. Take $\ltimes_j = 1 - 2^{-j}$. Since the sequence $(\ltimes_j)$ is an interpolating sequence for H^∞, there exists $\delta > 0$ such that

$$\prod_{k\neq j} \left|\frac{\ltimes_k - \ltimes_j}{1 - \overline{\ltimes}_k \ltimes_j}\right| \geq \delta$$

for all j. Thus, if $\mathcal{A}$ denotes the Blaschke product with zeros $\ltimes_1, \cdots\cdots, \ltimes_N$, then

$$|\mathcal{A}'(\ltimes_j)| \geq \frac{\delta}{1 - |\ltimes_j|^2} \ \gtrsim \ 2^j$$

for $j = 1, \cdots\cdots, N$. It follows that $\|\mathcal{A}'_\rho\|_\infty \geq 2^j$ for $\rho \geq \ \ltimes_j$ and thus

$$\|\mathcal{A}\|^{\frac{1}{2}+3\epsilon}_{\mathcal{A}^0_{\infty,\frac{1}{2}+3\epsilon}} \gtrsim \sum_{j=1}^{N-1} \int_{\ltimes_j}^{\ltimes_j+1} (1-\rho)^{-\frac{1}{2}+3\epsilon} \ \|\mathcal{A}'_\rho\|_\infty^{\frac{1}{2}+3\epsilon} d\rho \gtrsim \ N.$$

4. Conclusion

The conclusion reached by the study shows that mathematics is one of the sciences that deals with complex problems in life, and how to differentiate and choose the most appropriate of the variables that arise in your lif,,e, and therefore we use functions that govern the relationship of variables with each other using standard notation, and the conclusion we reached for two positive comparable functions, and symbolizes all acceptable variables with a symbol with a constant, and that there is a function that dominates a function, with proof that the sequence is almost approaching the upper limit by determining three regions of Nikolsky, and therefore improve the upper limit to give a different behavior for this quantity, it is the best possible estimate of our growth smaller than the estimate for a group of parameters. The analytical capabilities of finite sequences in unitary discs in zero-smooth Besov spaces, for a set of parameters, are optimal. The work is motivated from the perspective of complex analysis by the description of zero/singularity groups.

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